Morita Theory

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Abstract

Morita theory is the study of rings via their modules, which provides a weaker notion of sameness than isomorphism. This is done using category theory, which is the language of modern pure mathematics; it focuses on global properties rather than local properties, allowing for the relation of seemingly different mathematical objects.

The goal of this project is to understand how Morita theory gives an equivalence of categories between modules over a ring and modules over the endomorphism ring of a generator, where this second category is often more explicit and easier to understand.

1 Background Knowledge

Modules over a ring are a generalisation of the notion of vector spaces over a field. The definition of a module is similar to that of a vector space, and just as vector spaces often reveal properties of their underlying fields, we can often learn a lot about a ring by studying its modules.

Definition. *Let R be a commutative ring. An R-*module *consists of an abelian group M together with a map*

$$R \times M \longrightarrow M,$$

written as $(r, m) \longmapsto rm$, satisfying the following axioms:

- *for all $m \in M$, we have $1m = m$;*
- *for all $r, s \in R$ and $m \in M$, we have $(rs)m = r(sm)$;*
- *for all $r, s \in R$ and $m \in M$, we have $(r + s)m = rm + sm$;*
- *for all $r \in R$ and $m, n \in M$, we have $r(m + n) = rm + rn$.*

We can also define left and right R-modules over a non-commutative ring R, and a bimodule is a module that is both a left and a right module, such that the left and right multiplications are compatible.

Given a commutative ring R, and two R-modules M and N, we can define a new R-module $M \oplus N$, the *direct sum* of M and N, to consist of the abelian group $M \times N$ together with the following definition of multiplication:

$$r(m, n) = (rm, rn) \quad (\textit{for } r \in R,\ m \in M \textit{ and } n \in N).$$

If we do this repeatedly on M, forming $M \oplus \ldots \oplus M$, we get M^n for some n. An element of M^n can be thought of as an n-tuple $(m_1, \ldots, m_n)$ of elements of M. If M happens to be R itself we say that the resulting module R^n is a *free* module of rank n over R.

An *R-module homomorphism* $f : M \longrightarrow N$ consists of a homomorphism of abelian groups from M to N with the property that $f(rm) = rf(m)$ for all $r \in R$ and $m \in M$.

We say that elements $m_1, \ldots, m_k \in M$ of a module M generate M if, for all $m \in M$ there are $r_1, \ldots, r_k \in R$ such that

$$r_1 m_1 + \ldots + r_k m_k = m.$$

If there are finitely many elements $m_1, \ldots, m_k$ which generate M, we say that it is *finitely generated*. It is clear that the free module R^n is finitely generated.

Given a ring R and two R-modules M and N we also have a way to combine them using their tensor product $M \otimes_R N$.

Definition. *Let R be a ring, M a right R-module, and N a left R-module. The* tensor product *over R*

$$M \otimes_R N$$

is an abelian group together with a bilinear map

$$\otimes : M \times N \longrightarrow M \otimes_R N$$

satisfying the following universal property:

for every abelian group G and every bilinear map

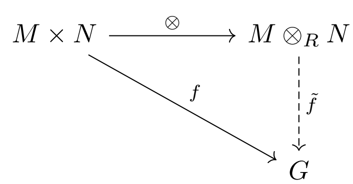
$$f : M \times N \longrightarrow G$$

there is a unique *group homomorphism*

$$\tilde{f} : M \otimes_R N \longrightarrow G$$

such that

$$\tilde{f} \circ \otimes = f.$$



This definition has the important consequence that every element of $M \otimes_R N$ has the form

$$\sum_{i=1}^{k} m_i \otimes n_i$$

for $m_i \in M$, $n_i \in N$, and there are relations

$$m_1 + rm_2 \otimes n = m_1 \otimes n + rm_2 \otimes n,$$
$$m \otimes (n_1 + rn_2) = m \otimes n_1 + rm \otimes n_2.$$

A category is a system of mathematical objects, with some notion of morphisms between objects relating them to one another. Some examples of objects are groups and topological spaces, with morphisms being homomorphisms and continuous maps respectively.

Definition. *A* category $\mathscr{A}$ *consists of:*

- *a collection* $ob(\mathscr{A})$ *of* objects*;*
- *for each* $A, B \in ob\,(\mathscr{A})$ *a collection* $\mathscr{A}(A, B)$ *of* morphisms *from* A *to* B*;*
- *for each* $A, B, C \in ob(\mathscr{A})$*, a function*

$$\begin{aligned}\mathscr{A}(B, C) \times \mathscr{A}(A, B) &\longrightarrow \mathscr{A}(A, C) \\ (g, f) &\longmapsto g \circ f,\end{aligned}$$

  called composition*;*
- *for each* $A \in ob(\mathscr{A})$*, an element* 1_A *of* $\mathscr{A}(A, A)$ *called the* identity *on* A*,*

satisfying the following axioms:

- associativity: *for each* $f \in \mathscr{A}(A, B)$*,* $g \in \mathscr{A}(B, C)$ *and* $h \in \mathscr{A}(C, D)$*, we have* $(h \circ g) \circ f = h \circ (g \circ f)$*;*
- identity laws*: for each* $f \in \mathscr{A}(A, B)$*, we have* $f \circ 1_A = f = 1_B \circ f$*.*

A morphism $f : X \longrightarrow Y$ is:

- a *monomorphism*, or *monic*, if for all morphisms $g_1, g_2 : Z \longrightarrow X$,

$$f \circ g_1 = f \circ g_2 \Longrightarrow g_1 = g_2;$$

- an *epimorphism*, or *epic*, if for all morphisms $g_1, g_2 : Y \longrightarrow Z$,

$$g_1 \circ f = g_2 \circ f \Longrightarrow g_1 = g_2;$$

- an *isomorphism* if there exists a morphism $g : Y \longrightarrow X$ such that $gf = 1_X$ and $fg = 1_Y$;
- an *endomorphism* if $X = Y$.

Categories themselves are a mathematical object, so unsurprisingly there is a sensible notion of morphisms between them. These morphisms are called functors.

Definition. *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be categories. A* functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ *consists of:*

- *a function*

$$ob(\mathscr{A}) \longrightarrow ob(\mathscr{B}),$$

  written as $A \longmapsto F(A)$*;*
- *for each* $A, A' \in \mathscr{A}$*, a function*

$$\mathscr{A}(A, A') \longrightarrow \mathscr{B}(F(A), F(A')),$$

  written as $f \longmapsto F(f)$*,*

satisfying the following axioms:

- $F(f' \circ f) = F(f') \circ F(f)$ *whenever* $A \xrightarrow{f} A' \xrightarrow{f'} A''$ *in* $\mathscr{A}$*;*

- $F(1_A) = 1_{F(A)}$ *whenever* $A \in \mathscr{A}$.

More surprisingly, we can also define morphisms between functors, which are known as natural transformations. This only applies when the functors have the same domain and codomain.

Definition. *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be categories and let* $\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B}$ *be functors. A* natural transformation $\alpha : F \longrightarrow G$ *is a family* $\left(F(A) \xrightarrow{\alpha_A} G(A)\right)_{A \in \mathscr{A}}$ *of maps in* $\mathscr{B}$ *such that for every map* $A \xrightarrow{f} A'$ *in* $\mathscr{A}$*, the square*

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_{A'}} \\ G(A) & \xrightarrow{G(f)} & G(A') \end{array}$$

commutes. The maps α_A *are called the* components *of* A*.*

An equivalence between two categories is a pair of functors between them which are inverse to each other up to natural isomorphism of functors. We say that two categories are equivalent if there is an equivalence between them. A functor exhibits an equivalence of categories precisely when it is both essentially surjective (surjective up to isomorphism) and fully faithful (both injective and surjective when restricted to each set of morphisms that have a given source and target).

The categories we are primarily interested in are categories of modules over rings. Given a ring R, the *category of left modules* over R is the category whose objects are all left modules over R and whose morphisms are all module homomorphisms between left R-modules. The *category of right modules* is defined in a similar way.

Given a ring homomorphism $\theta : R \longrightarrow S$, there are three functors between R-mod and S-mod which change the coefficient ring of a module. These can be described as follows:

$$R\text{-Mod} \;\overset{S\otimes_R -}{\longrightarrow}\; S\text{-Mod}, \qquad R\text{-Mod} \;\overset{\theta^*}{\longleftarrow}\; S\text{-Mod}, \qquad R\text{-Mod} \;\underset{\mathrm{Hom}_R(S,-)}{\longrightarrow}\; S\text{-Mod}$$

- Restriction of scalars $\theta^\star$:

$$\begin{aligned} R \times M &\longmapsto M \\ (r, m) &\longmapsto \theta(r) \cdot m \end{aligned} \qquad\qquad \begin{aligned} f : M \to N &\longmapsto M \to N \\ & \qquad m \to f(m) \end{aligned}$$

- Extension of scalars $S \otimes_R -$:

$$\begin{aligned} S \times (S \otimes_R M) &\longmapsto S \otimes_R M \\ (s', s \otimes m) &\longmapsto s's \otimes m \end{aligned} \qquad\qquad \begin{aligned} f : M \to N &\longmapsto S \otimes_R M \to S \otimes_R N \\ & \qquad s \otimes m \to s \otimes f(m) \end{aligned}$$

- Coextension of scalars $\mathrm{Hom}_R(S, -)$:

$$S \times \mathrm{Hom}_R(S, M) \longmapsto \mathrm{Hom}_R(S, M) \qquad h : M \to N \longmapsto \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, N)$$
$$(s', f : s \to m) \longmapsto g : s \to f(s's) \qquad f : s \to f(s) \to g : s \to hf(s)$$

2 Morita Equivalence

The term 'Morita theory' is usually used for results concerning equivalences of various kinds of module categories. These results can be extended to various different contexts, such as abelian categories, derived categories and stable model categories, but we focus on modules over rings.

It turns out that modules over a ring are characterized by the existence of a 'small generator', which plays the role of the free module of rank one, and that this concept of a 'small generator' continues to be relevant in more and more complex kinds of module category.

First, we review some important definitions.

Definition. *Let R be a ring and M an R-module.*

- *M is* small *if the functor $Hom_R(M, -)$ preserves sums; that is, the map*

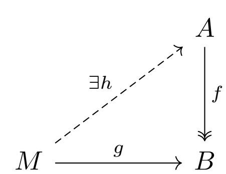
$$Hom_R(M, \oplus A_i) \overset{\cong}{\longleftarrow} \bigoplus Hom_R(M, A_i)$$

  is always an isomorphism.

- *M is* projective *if the following equivalent conditions hold:*
  - *M is a* direct summand *of a free module, i.e. there exists an R-module A such that $M \oplus A$ is free.*
  - *$Hom_R(M, -)$ is an* exact *functor, i.e. it preserves surjective maps.*

$$f\colon A \twoheadrightarrow B \Longrightarrow Hom_R(M, A) \twoheadrightarrow Hom_R(M, B)$$

  - *For every surjective homomorphism $f\colon A \longrightarrow B$ and homomorphism $g\colon M \longrightarrow B$, there exists a* lifting homomorphism *$h : M \longrightarrow A$ such that $f \circ h = g$.*



- *M is a* generator *if for any R-module N, there exists a surjective map from N to a direct sum of (possibly infinitely many) copies of M.*

Our definition of smallness is mainly relevant when considering infinite sums of modules, since finite sums of modules are isomorphic to finite products, and so are automatically preserved by the hom functor. For modules over a ring, smallness is closely related to the idea of finite generation.

Lemma. *Every finitely generated module is small.*

Proof. It is easy to see that the free module R^n is small; R is clearly small, and if A and B are both small, then so is $A \oplus B$, so R^n is small by induction on n.

We also have that if an R-module M is finitely generated, there exists a surjective homomorphism of R-modules

$$\phi : R^n \longrightarrow M$$

for some finite n. If M is generated by $m_1, \ldots, m_n \in M$, we define this R-module homomorphism via

$$\phi(a_1, \ldots, a_n) = a_1 m_1 + \ldots + a_n m_n.$$

We can see that ϕ is a homomorphism:

$$\phi(ra_1, \ldots, ra_n) = ra_1 m_1 + \ldots + ra_n m_n = r\phi(a_1, \ldots, a_n)$$
$$\phi(a_1 + b_1, \ldots, a_n + b_n) = (a_1 + b_1)m_1 + \ldots + (a_n + b_n)m_n = \phi(a_1, \ldots, a_n) + \phi(b_1, \ldots, b_n)$$

Since $m_1, \ldots, m_n$ are generators, ϕ is surjective.

All that remains is to show that if an R-module M is small and we have a surjective homomorphism $q : M \twoheadrightarrow N$, then N is small. Let

$$p_j : \oplus_i B_i \longrightarrow B_j$$

be the obvious projection ($p_j \circ i_j = 1_{B_j}$, else $p_j \circ i_k = 0$), and given $f : N \longrightarrow \oplus_i B_i$, put

$$f_j := p_j \circ f.$$

Then N is small if for each $f : N \longrightarrow \oplus_i B_i$, we have $f_j = 0$ for all but finitely many j.

But for any $f : N \longrightarrow \oplus_i B_i$, we have $p_j \circ f \circ q = 0$ for all but finitely many j, so $f_j = p_j \circ f = 0$ since q is surjective. □

It turns out that if a projective module is small, then it is finitely generated, but this is not true in the general case.

We are now ready to prove the classical Morita theorem for rings, as stated below.

Theorem. *For two rings R and S, the following conditions are equivalent.*

1. *The categories of right R-modules and right S-modules are equivalent.*
2. *The category of right S-modules has a small projective generator whose endomorphism ring is isomorphic to R.*
3. *There exists an R-S-bimodule M such that the functor*

$$- \otimes_R M : \textit{Mod-R} \longrightarrow \textit{Mod-S}$$

is an equivalence of categories.

If these conditions hold, then R and S are said to be Morita equivalent.

Proof. First, suppose (1) holds, and let

$$F : \text{Mod-}R \longrightarrow \text{Mod-}S$$

be an equivalence of categories. R is a small projective generator of the category of R-modules, as it is the free module of rank one. Being small, projective or a generator are categorical properties, so they are preserved by an equivalence of categories. Thus, the S-module FR is a small projective generator of the category of S-modules.

Since F is an equivalence of categories, in particular it is a fully faithful functor. So we have isomorphisms of rings

$$F : R \cong \text{End}_R(R) \xrightarrow{\cong} \text{End}_S(FR).$$

Now suppose (2) holds, and let P be a small projective S-module which generates the category Mod-S. By assumption, we have an isomorphism $f : R \cong \text{End}_S(P)$. As such, we can view P as an R-S-bimodule by setting $r \cdot x = f(r)(x)$ for $r \in R$ and $x \in P$.

We can then show that P satisfies the conditions of (3) by finding a pair of inverse equivalences between the module categories.

First, we have the R-linear map

$$X \longrightarrow \text{Hom}_S(P, X \otimes_R P), \quad x \longmapsto (y \longmapsto x \otimes y).$$

For $X = R$, this map coincides with the isomorphism f, so it is bijective. Since P is small, both sides commute with both finite and infinite sums, so the map is bijective for every free R-module. Since P is projective over S, both sides are right exact as functors of X. Every R-module is the cokernel of a morphism between free R-modules, so the map is bijective in general.

Second, we have the S-linear evaluation map

$$\text{Hom}_S(P, Y) \otimes_R P \longrightarrow Y, \quad \phi \otimes x \longmapsto \phi(x).$$

For $Y = P$, the right action of R on $\text{Hom}_S(P, P)$ arises from the isomorphism $R \cong \text{Hom}_S(P, P)$, so the map is bijective. Now, as above, both sides are right exact and commute with sums, finite or infinite. Since P is a generator, every S-module is the cokernel of a morphism between direct sums of copies of P, so the map is bijective in general.

Condition (1) trivially follows from (3), so we are done. □

Example. *The simplest example of Morita equivalence comes from linear algebra, as it follows from the way we define matrices. We have that any free R-module of finite rank $n \geq 1$ is a small projective generator for the category of right R-modules. The ring of $n \times n$ matrices with entries in R*

$$M_n(R) \cong End_R(R^n)$$

is isomorphic to the endomorphism ring of R^n, as a matrix is a way to represent a linear transformation of vectors. Thus, R and $M_n(R)$ are Morita equivalent.

The bimodules inducing this Morita equivalence can be taken to be R^n, viewed as row vectors (i.e. $1 \times n$ matrices) and column vectors (i.e. $n \times 1$ matrices) respectively.

It is clear that any two isomorphic rings are Morita equivalent. In fact, the notion of Morita equivalence is of interest only when dealing with noncommutative rings, as if a ring is commutative the two properties always coincide.

Lemma. *Two commutative rings are Morita equivalent if and only if they are isomorphic.*

Proof. We can show that the center $Z(R) = \{r \in R \mid rs = sr \text{ for all } s \in R\}$ is Morita invariant, since the center is isomorphic to the endomorphism ring of the identity functor of Mod-R. It turns out that the endomorphisms $x \longmapsto xc,\ x \in R,\ c \in Z(R)$ are the only endomorphisms $h(c)$ of the identity functor. We can construct the corresponding ring isomorphism

$$h : Z(R) \longrightarrow \mathrm{End}(Id_{\mathrm{Mod}_R})$$

and show that it is in fact an isomorphism as follows. First, since $h(c)_R(1) = 1 \cdot c = c$ it follows that h is injective. It is clearly a ring homomorphism. In order to prove that h is surjective, let $t \in \mathrm{End}(Id_{\mathrm{Mod}_R})$ be a natural transformation on the identity functor, and let $c = t_R(1)$. By naturality of t,

$$\begin{array}{ccc} A & \xrightarrow{\ t_A\ } & A \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f} \\ M & \xrightarrow{\ t_M\ } & M \end{array}$$

commutes, so $t_M(x) = t_M(f(1)) = f(t_R(1)) = f(c) = xc = h(c)_M(x)$. Thus $t = h(c)$, and h is therefore surjective.

Due to this equivalence, the center of any ring is invariant under Morita equivalence. In particular, for a commutative ring R we have that $Z(R) = R$, so any two Morita equivalent rings have isomorphic centers and are thus isomorphic. □

Finally, we can look at a particular type of modules called *torsion modules*, and see what we can learn about them with the tools given to us by Morita theory.

Definition. *An element m of a module M over a ring R is called a* torsion element *of the module if there exists a regular element r of the ring which has a power that annihilates m, i.e. $r^n m = 0$ for some n. A module M is called a* torsion module *if all its elements are torsion elements.*

Given an ideal I of a ring R, we say that a module is I-torsion if every element $i \in I$ has some power which annihilates every element $m \in M$. Due to results [2] in more advanced contexts such as derived categories, one could speculate that

$$I\text{-torsion-}R\text{-mod} \simeq \text{Mod-}\mathscr{E},$$

where

$$\mathscr{E} = \mathrm{End}_R(R/I).$$

However, it is in fact the case that we need a stronger condition for this statement to be true. Define an I-primary torsion module to be a module where for every element $i \in I$ and $m \in M$, $im = 0$. Then we can show the following.

Lemma. *An R-module can be made into an R/I-module if and only if it is I-primary torsion.*

Proof. Let M be an R/I-module. Then $r \cdot m = (r + i) \cdot m$ for any $i \in I$, $r \in R$ and $m \in M$, so $r \cdot m = r \cdot m + i \cdot m$, so $i \cdot m = 0$, and thus M is I-primary torsion.

For the other direction, suppose M is I-primary torsion. Define an action

$$\begin{aligned} R/I \times M &\longrightarrow M \\ (\bar{r}, m) &\longmapsto rm \end{aligned}$$

This certainly satisfies the module axioms, so we just need to verify that the action is well defined. Suppose $\bar{r} = \bar{r'}$, i.e. $r = r' + i$ for some $i \in I$. We need $rm = (r' + i)m = r'm + im$. But this is true, since by assumption $im = 0$. So M is an R/I-module with this action. □

Recall that an integral domain is a nonzero commutative ring in which the product of any two nonzero elements is nonzero. If we happen to be working in an integral domain, then this reduces to the case we were originally hoping for.

Lemma. *In an integral domain R, if a module M is I-torsion for some ideal I then it is also I-primary torsion.*

Proof. Assume that M is I-torsion. Then for any $m \in M$ and $i \in I$, we have that $i^n m = 0$ for some n. Since R is an integral domain, if $i^n m = 0$ then either $i^n = 0$ or $m = 0$. If $m = 0$, then $im = 0$ and we are done. If $i^n = 0$, then $i = 0$ by induction on n, so in this case $im = 0$ also. Thus, M is I-primary torsion. □

In particular, this means that in any field F I-torsion is equivalent to I-primary torsion, since every field is an integral domain.

It is possible to show that the equivalence between modules with I-primary torsion and R/I-modules is in fact an equivalence of categories, i.e.

$$I\text{-primary torsion-}R\text{-mod} \simeq R/I\text{-mod},$$

and that by the application of Morita theory this gives an isomorphism

$$I\text{-primary torsion-}R\text{-mod} \cong \text{Mod-}\mathscr{E},$$

where

$$\mathscr{E} = \text{End}_R(R/I).$$

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